Problem 4A,4
show that the constant 3 in the Vitali Covering cannot be replaced by a smaller positive constant.

Proof: Just take two intervals, $(0,1),(1-\frac{1}{n},2-\frac{1}{n})$, where n can be arbitrary large positive integer.

Problem 4A,8
Find a formula for the Hardy-Littlewood maximal function of the function $h:\mathbb{R}\to[0,\infty)$ defined by

$$f(x)=\begin{cases} x & if\ 0\le x\le 1\\ 0 & otherwise\end{cases}$$

Proof:

$$f^*(x)=\begin{cases} \dfrac{1}{4(1-x)} & if\ x\in(-\infty,\dfrac{1}{2}]\\ x & if\ x\in[\dfrac{1}{2},1)\\ \dfrac{x-\sqrt{x^2-1}}{2} & if\ x\in(1,\infty)\end{cases}$$

Problem 4A,11
Give an example of a Borel measurable function $h:\mathbb{R}\to[0,\infty)$ such that $h^*(b)<\infty$ for all $b\in\mathbb{R}$ but $\sup\{h^*(b)|b\in\mathbb{R}\}=\infty$.

Proof: We construct a continuous integrable function f as follows: for each integer n, set $f(n)=|n|, f(n-\frac{1}{n^9})=0$, $f(n+\frac{1}{n^9})=0$. Then let f be linear function between these points. Since $\sum_{n=-\infty}^{\infty}\frac{1}{n^8}<\infty$, we know the function is integrable and continuous. Note $f^*(x)\ge f(x)$, we know that $\sup\{h^*(b)|b\in\mathbb{R}\}=\infty$. Note that for every $x_0\in\mathbb{R}, \lim_{b\to+\infty}\frac{1}{2b}\int_{x_0-b}^{x_0+b}f(x)dx=0, \lim_{b\to 0}\frac{1}{2b}\int_{x_0-b}^{x_0+b}f(x)dx=f(x_0), \frac{1}{2b}\int_{x_0-b}^{x_0+b}f(x)dx=0$ is continuous function in $b\in(0,\infty)$, we know that $h:\mathbb{R}\to[0,\infty)$ such that $h^*(b)<\infty$ for all $b\in\mathbb{R}$.

Problem 4A,13
Show that there exists $h\in\mathcal{L}^1(\mathbb{R})$ such that $h^*(b)=\infty$ for every $b\in\mathbb{Q}$.

Proof: First we can construct a integrable function g such that $g^*(0)=\infty, g^*(x)<\infty$ for each $x\ne 0$ (e.g. consider $g(x)=\log|x|$ near 0). Now let $\{q_k\}_{k=1}^{\infty}$ be all rational numbers. Our h can be chosed to be

$$h(x)=\sum_{k=1}^{\infty}\frac{1}{2^k}g(x-q_k).$$

It is easy to check this is convergent series and satisfies all the assumption.

Problem 4B,2
Suppose $f\in\mathcal{L}^1(\mathbb{R})$.Prove that for almost every $b\in\mathbb{R}$,

$$\lim_{t\downarrow 0}\sup\{\frac{1}{|I|}\int_I|f-f_I|: I\ is\ an\ interval\ of\ length\ t\ containing\ b\}=0.$$

Proof: Let b be a Lebesgue point, which means

$$\lim_{t\downarrow 0}\frac{1}{2t}\int_{b-t}^{b+t}|f-f(b)|=0.$$

Let I be an interval containing b. Then $|\frac{1}{|I|}\int_I f(x)dx - f(b)| = |\frac{1}{|I|}\int_I (f(x)-f(b))| \le \frac{1}{|I|}\int_I |f(x)-f(b)|dx$ $\le \frac{1}{|I|}\int_{b-|I|}^{b+|I|}|f(x)-f(b)|dx$. Therefore

$$\lim_{t\downarrow 0}\sup\{f_I - f(b) : I \textit{ is an interval of length } t \textit{ containing } b\} = 0$$

This obviously implies the

$$\lim_{t\downarrow 0}\sup\{\frac{1}{|I|}\int_I |f-f_I| : I \textit{ is an interval of length } t \textit{ containing } b\} = 0.$$

Problem 4B,3
Suppose $f:\mathbb{R}\to\mathbb{R}$ is a Lebesgue measurable function such that $f^2 \in \mathcal{L}^1(\mathbb{R})$. Prove that

$$\lim_{t\downarrow 0}\frac{1}{2t}\int_{b-t}^{b+t}|f-f(b)|^2 = 0$$

Proof: Note that $f^2 \in \mathcal{L}^1(\mathbb{R})$ implies $f \in \mathcal{L}^1_{loc}(\mathbb{R})$. And since we are dealing with local property, we in fact can assume $f \in \mathcal{L}^1(\mathbb{R})$. Then take b to be both Lebeges point of f, f^2. Then

$$\lim_{t\downarrow 0}\frac{1}{2t}\int_{b-t}^{b+t}|f-f(b)| = 0.$$

$$\lim_{t\downarrow 0}\frac{1}{2t}\int_{b-t}^{b+t}|f^2-f^2(b)| = 0.$$

which of course implies

$$\lim_{t\downarrow 0}\frac{1}{2t}\int_{b-t}^{b+t} f dx = f(b)$$

$$\lim_{t\downarrow 0}\frac{1}{2t}\int_{b-t}^{b+t} f^2 = f^2(b).$$

Then rewriting $\frac{1}{2t}\int_{b-t}^{b+t}|f-f(b)|^2 = \frac{1}{2t}\int_{b-t}^{b+t}f^2 - 2f(b)\frac{1}{2t}\int_{b-t}^{b+t}f + f^2(b)$, this completes the proof.

Problem 4B,9
Prove that if $t\in[0,1]$, then there exists a Borel set $E\subset\mathbb{R}$ such that the density of E at 0 is t.

Proof: We only need to consider $t\in(0,1)$. E can be chosen as follows:

$$E_1 = \bigcup_{n=1}^{\infty}(\frac{1}{n} - t(\frac{1}{n}-\frac{1}{n+1}), \frac{1}{n})$$

$$E_2 = \bigcup_{n=1}^{\infty}(-\frac{1}{n}, -t(\frac{1}{n+1}-\frac{1}{n}) - \frac{1}{n})$$

$$E = E_1 \bigcup E_2$$

Then it is easy to check this E satisfies the conclusion.